DOCUMENT RESUME

ED 476 632 SE 067 903

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TITLE	Beliefs about Proof in Collegiate Calculus.
PUB DATE	2001-00-00
NOTE	10p.; In: Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (23rd, Snowbird, Utah, October 18-21, 2001). p595-603. For full proceedings, see SE 065 231.
AVAILABLE FROM	ERIC/CSMEE Publications, 1929 Kenny Road, Columbus, OH 43210-1080. Tel: 800-276-0462 (Toll Free); Tel: 614-292-5680.
PUB TYPE	Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE	EDRS Price MF01/PC01 Plus Postage.
DESCRIPTORS	*Calculus; Higher Education; Knowledge Base for Teaching; Learning Problems; Mathematical Logic; Mathematics Instruction; *Mathematics Teachers; Problem Solving; *Thinking Skills

ABSTRACT

The broad aim of this research is to characterize the views of proof held by college calculus students and their two types of teachers — mathematics graduate students and professors. The analysis is based on an examination of the ways in which people in all three groups produce and evaluate different types of solutions to a proof-based problem from a college calculus course. Initial results indicate a subtle but fundamental difference in the way university level teachers and students view proof. To the teachers a proof is the connection between an idea and the representation of the idea - the representation used being a function of the norms of a particular mathematical community. To the students, who have not had many mathematical experiences outside of school mathematics and who may not understand the underlying mathematics, the proof is simply the representation. (Author)

BELIEFS ABOUT PROOF IN COLLEGIATE CALCULUS

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Abstract: The broad aim of this research is to characterize the views of proof held by college calculus students and their two types of teachers—mathematics graduate students and professors. The analysis is based on an examination of the ways in which people in all three groups produce and evaluate different types of solutions to a proof-based problem from a college calculus course. Initial results indicate a subtle but fundamental difference in the way university level teachers and students view proof. To the teachers a proof is the connection between an idea and the representation of that idea—the representation used being a function of the norms of a particular mathematical community. To the students, who have not had many mathematical experiences outside of school mathematics and who may not understand the underlying mathematics, the proof is simply the representation.

Introduction

The aim of this research is to characterize the views of proof held by college calculus students and their two types of teachers—mathematics graduate students and professors. I do so by examining the ways in which people in all three groups produce and evaluate different types of solutions to a proof-based problem from a college calculus course. Most work on proof at the university level has focused largely, if not entirely, on students. The results of those studies have yielded many interesting insights, one of which is to suggest that the views of proof are different from—and possibly even in conflict with—the views held by their teachers (e.g., Alibert & Thomas, 1991; Harel & Sowder, 1998). However, those studies by themselves fall short of determining if—and if so, *how*—those views conflict, because they only hypothesize about the views held by mathematicians. The study reported here is designed to address these questions, using a data collection method similar to one developed to compare student and teacher views of proof at the high school level (Hoyles & Healy, 1999), but on a smaller scale and with more in-depth interviews.

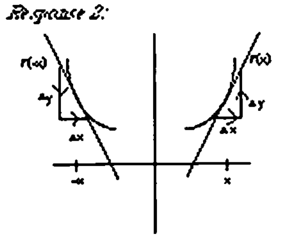
This study is situated within a broad literature indicating that students' mathematical difficulties are not only *cognitive*, e.g., they do not connect concept images with concept definitions (Vinner, 1991), but also *epistemological*, e.g., their view of what constitutes knowing may affect how they reason (Hofer, 1994), and *social,* e.g., the kind of mathematics arguments they generate are constrained by their expectations of school mathematics (Balacheff, 1991; Schoenfeld, 1992). Recent research has indicated that cognitive, epistemological, and social factors are related. For instance, certain perceptions about the nature of proof (e.g, what is the appropriate level of rigor) lead to difficulty in producing a proof (e.g. in an exam situation) (Moore, 1994). And some beliefs, such as what kind of answer is expected on an exam, may conflict with others, such as what kind of answer would demonstrate the best understanding (Elby, 1999). Continuing this line of research, an overarching goal of this research project to better understand the role of cognitive, epistemological, and social factors in shaping people's views of proof.

Methods

Using a task-based interview protocol, 11 students, 4 mathematics graduate students, and 5 mathematics professors were individually interviewed. Each interview lasted 1-1.5 hours and were both audio and videotaped. The students came from three different sections of a first semester calculus course at a top-ranked university. The calculus course was traditional, in the sense that there was a strong emphasis on rigor (the textbook was Stewart (1998)) and lectures were closely aligned with the textbook. The professors who gave the lectures and graduate students who led discussion sections were among the subjects.

The central question for the interview was to *prove that the derivative of an even function is odd.* Participants were first asked to answer this question on their own. They were then shown different responses to the question and asked to evaluate them. Before seeing the responses, participants were asked to discuss their work, focusing on why they chose a particular method and how convinced they were of their response. Next they were shown five responses to the question, not all of which were correct, which came from pilot studies and textbooks. Response #1 was empirical (checking $y=x^n$ for n from 1 to 6), #2 was graphical, #3 was a textbook-like proof using the definition of derivative, #4 was a short proof using the chain rule, and #5 was a false formal-looking proof. In much of rest of the paper I focus on people's views about #2 and #3 so they are reproduced in Figure 1.

Response 2:



If f(x) is an even function it is symmetric over the y axis. So the slope at any point x is the opposite of the slope at -x. In other words f '(-x) = - f '(x), which means the derivative of the function is odd.

Response 3:

Want to show if f(x) = f(-x) then - f '(x) = f '(-x).

f '(-x) = f '(x) by the definition of derivative

$= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$ since f is even

Let t=-h

$= \lim_{t \to 0} \frac{f(x+t) - f(x)}{-t}$

$= - \lim_{t \to 0} \frac{f(x+t) - f(x)}{t}$

= -f '(x) as desired.

Figure 1.

Participants were asked to judge each response based on different criteria such as: (1) Is it convincing? Why or why not?, (2) How many points would this get on an exam? Why? (3) What response (or parts or combinations of responses) do you prefer? Why?, and (4) What response (or parts or combinations of responses) would demonstrate the best understanding? Why?

A second round of interviews was conducted to address issues of validity and scope that arose after a preliminary analysis of the first round data: how reliable are the professed beliefs, how typical are the beliefs/understandings of the individuals in the study, and how well do the five responses shown to people in the study provide access to people's beliefs and understandings about proof? A sample of participants from the first round (1 professor, 1 graduate student, and 2 undergraduates) was chosen from those who volunteered to be interviewed again. The participants were interviewed one semester after they were originally interviewed. They were shown some of the results of the study (which included comments from un-named participants in each group about each of the responses) and were asked whether the views expressed in those comments seemed typical of view of people in each of the three groups, and whether the views expressed in those comments reflected his or her own personal views.

Results

To get a rough sense of how student and teacher views compare, we can look at which responses they found deserved full credit (Figure 2), and which demonstrated the best understanding (Figure 3).

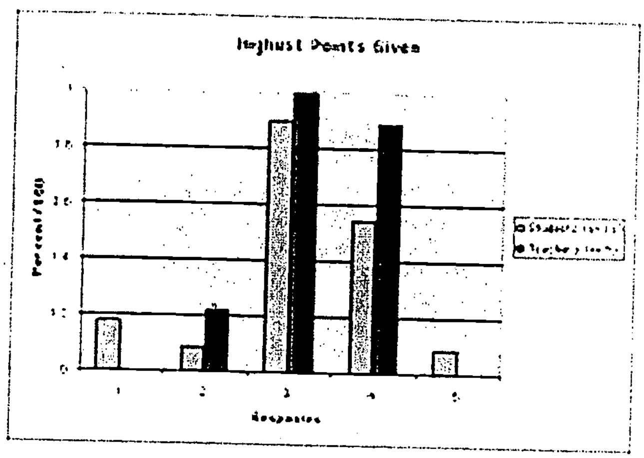


Figure 2.

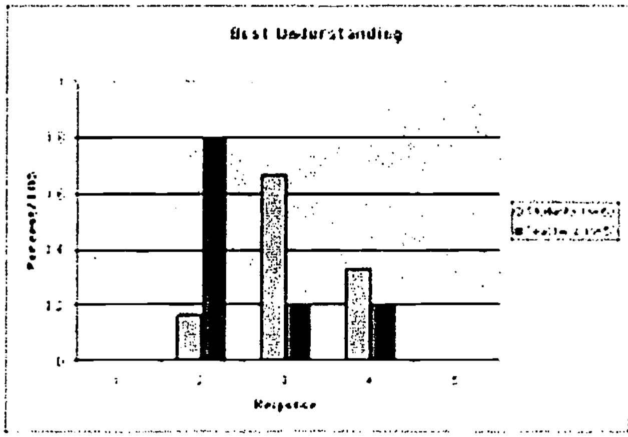


Figure 3.

Although both the students and faculty gave the textbook-proof (#3) full credit, they did so for different reasons. The students tended to like #3 because it stated the definition of derivative and used formal language.

Student CH[1]: I like that one (#3). Yeah, when I was doing the proof, I was trying to think how I could use limits. I couldn't really see how I would do the odd functions and the even functions, but yeah, this looks like something he would take as a proof.

Students thought #3 deserved full credit even if they admitted that they didn't completely understand it.

Student KY: So they proved it (#3) I think. I would give them full credit, 10 out of 10.

I: And what would your professor give?

Student KY: I also think the professor would give 10 out of 10. As long as he understands a little bit more, because I got lost a little bit.

Both students and teachers seemed to recognize #3 as the type of proof that could appear in a textbook. The students seemed to think a proof looking like a textbook-proof made it a better response than #4 or #2.

Student AN: The response 3, I gave it full credit because it makes sense. Actually I like a proof like that because it derives from something we know and gets to the point what we want. And its very useful for me to look at it. I get it right away. [...] I'll remember it. And this is something that deserves credit, deserves full credit because that is how the textbook does and it is really easy for a person to recognize it too. It is really useful.

Several students went so far as to say the proof was perfect. However, several teachers felt that #3 was not an ideal response.

I: And what did you think about response #3?

Prof A: (laugh) Wildly too... Way too complicated, but correct in every detail. You get points for correctness even if it is not the shortest proof. [...] This is like Stewart. A long proof of something very simple.

Several people commented on some combination of #2 and #3 being preferable in some way, but again the reasons differed between students and teachers. For the teachers, #2 and #3 seemed to be saying roughly the same thing, differing only in the amount of rigor. The combination seemed to provide a combination of understanding and rigor, with the algebraic proof being more general or rigorous and the picture proof supplying a sense of understanding.

Prof B: Well, the problem from my standpoint is that it (#2) is not a proof. If I were going to use that picture, I would take it and turn it into a proof. Although if you do that, it comes down to pretty much this (#3). The two together might be the best proof from the standpoint of the students. The problem with this (#3) is that the manipulations are not transparent to them. This (#2) is a lot more transparent. The two together may make maybe a good proof for them.

Some students also saw similarities between #2 and #3. However, there was a general tendency among students to value the algebra, not just because it is more general or rigorous, but also because it, more than the picture, provided a sense of why the claim was true.

I: Ok, so now looking back over all of them, what kind of response, or combination of responses, or something that is not even here would be the best if you wanted to really show that you understood this problem?

Student CH: Probably a combination of these two (#3 and 2) Drawing the graph, you know, of this particular function and then proving it you know based on this (#3). So then you have a visual that people can see and comprehend (based on #2). And then you give the reason why (based on #3). [...] I guess you could have added the two. Like, um, put the picture and then you know use the definition of the derivative and then stated that at the very end, combined them. But just the picture alone doesn't say much about the answer.

It is as if the picture is only of heuristic value; it says nothing about the truth of the claim, quite different from the teachers' point of view. Moreover, people's views about the algebraic proof appeared to be a function of understanding, as witnessed by this student who at first doesn't seem to understand #3 (and has a view similar to CH) and then seems to have an insight (and changes to a view more like that of Prof A):

Student KY: So in this case (#3), the understanding by a graph isn't necessary. I mean, not completely necessary.

I: Why do you say not completely necessary?

Student KY: Um, if you want to understand. If you, um...wow… yeah, man… I guess there are two different understandings. Like, what it does to the numbers, I guess and what it does to the picture.

[…]

I: Why do you say wow?

Student KY: I'm just seeing, I guess, the two different tracks you can take... the two different roads you can take when teaching this. I understand your studies I guess (laugh).

I: Can you say more about that?

Student KY: Well, I mean... its just that this (#3) seems more anal. And this (#2) seems a little bit more relaxed. But they both kind of show the same thing. More difficult (#3) . Less difficult (#2) . Um, um, hmm... I think if you want to go into math, I mean if you are a math major, both of them are necess… both of them.... I think you should know both of them. But this (#3) more so than this (#2).

Some teachers (especially graduate students) were reluctant to consider the textbook proof a complete proof, a picture being needed to make the proof complete.

Grad A: And if someone did this proof (#3) they would get full credit too. I might make a remark, like you could draw a picture too. So the true proof would be this (#3) and a picture. They would get extra brownie points or something.

At the same time, several teachers were reluctant to give a graphical proof full credit, because they did not think is was a real proof, or at least not a rigorous one. At least one graduate student expressed some angst over this reluctance.

Grad B: (About #2) I would like to give it full credit, but somehow I feel I'm just not allowed to. That a picture isn't good enough. It doesn't look like there is any math written down. I know, that is so stupid.

Several teachers commented on a difference between the type of proof expected in an informal situation such as office hours or a discussion section, and what they would expect on an exam.

Grad C: So, if it's the discussion section, I would definitely first try a couple of examples before actually launching into the main proof. Because sometimes proofs are proofs. They are fine, except they are not very illuminating. And a better way of actually convincing yourself that a result is actually true is to work hands on an example. Which might not give you a clue as to what the most general rigorous proof might be, but at least it convinces the student that ok, at least what we want to prove is correct as we can see by examples. So if it is the discussion section, I would first start maybe with a little bit of this and then actually ask how to prove it in the general situation. In an exam of course that is not going to be... that's the place where you just write down what is correct.

And some teachers (compared with no students) preferred the graphical proof over the textbook proof, not just for demonstrating understanding, but even as an answer on an exam.

Prof C: #2 I would simply accept, even if it is not rigorous. Somehow I think it is so useful that a student could think in these terms, I think, for his or her career, I think it is more important than their ability to write down these polynomials. So #2 I think I would simply accept without many questions.

Perhaps significantly, Prof R was the only professor in the study who had never taught freshman calculus (he almost exclusively teaches advanced graduate courses.)

The teachers distinguished between a formal proof which is not "very illuminating" (Grad C) and an intuitive proof which is more "transparent" (Prof B). This seems to indicate the existence of some idea, which in this case is more closely represented by a picture like in #2 than by #3. The intuitive proof is valued more for the purpose of convincing oneself of the a claim (ascertaining, in Harel and Sowder's language) and the formal proof is valued for its ability to establish the truth the language and level of rigor expected of a particular community (persuading, to Harel and Sowder). Consider the comments of this professor after he generated two different proofs for the odd/even question, one with a picture and one with the chain rule.

I: So now you have two different approaches. How do those compare in terms of how convinced you are?

Prof A: Oh, the first one (picture proof) convinces me completely that it is right, it is right. The second one (chain rule proof) is how you present it if you want to convince somebody else. It doesn't have… (sigh, look to side) your currency. My currency is kind of… my currency is like pictures. But the general currency that works for everybody is a formula.

This professor, by the way, preferred the chain rule proof on an exam to a picture because he thought it was easier to grade.

The teachers' comments seem to suggest that what they believe to be a true proof is the connection between the idea and the representation of that idea, but what counts as an appropriate representation is a function of their audience. Even though the teachers made distinctions between different types of proofs, they clearly see how they are related "If I were going to use that picture, I would take it and turn it into a proof. Although if you do that, it comes down to pretty much this (#3)" (Prof B). And some even went so far as to say the picture was necessary for the proof to be complete, "So the true proof would be this (#3) and a picture." (Stud KY) Many teachers favored an algebraic representation (Prof B, Grad B, Grad C) at least in the context of a course. However, I claim that what actually makes them call the algebraic proof (#3 or #4) a proof is that they can connect it to their idea of a proof.

In contrast, the students—many of which do not appear to really understand a proof like #3—are not able to connect the idea to the representation. Another way of saying they do not understand the proof is to say they do not have the idea, something that completely convinces them of the veracity of the claim. So they have only the formal algebraic proof to value as proof. " But just the picture alone doesn't say much about the answer." (Stud CH) So unlike the way in which teachers view the algebraic proof—that its authority derives from a connection to a main idea, to the students, the algebraic proof appears to stand alone, almost disconnected from the idea: "the understanding by a graph isn't necessary. I mean, not completely necessary." (Stud KY) This is a subtle, but I think fundamental, difference in perspective because while on the surface things may look the same (e.g. both value the algebraic proof on an exam) the surface behavior belies significant differences in not only how well people understand, but possibly also what they take understanding to be.

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Note

[1] Transcript conventions: [...] means break in the transcript, parenthetical remarks indicate gestures, mostly used to indicate which response the participant is referring to.